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Gravitating Instantons In 3 Dimensional Anti de Sitter Space

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Abstract

We study the Einstein-Chern-Simons gravity coupled to Yang-Mills-Higgs theory in three dimensional Euclidean space with cosmological constant. The classical equations of motion reduce to Bogomol'nyi type first order equations. There are BPS type instanton (monopole) solutions of finite action which we find numerically. In addition we point out to some exact solutions which are singular.

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1 Introduction

In recent years there has been a growing interest in the study of three dimensional gravity [1]. For example the BTZ [2] solution in three dimensions proved to be an extremely useful toy model to understand microscopic degrees of freedom of a black hole. Quite recently [3] a black hole with all three Abelian hairs (charge, angular momentum and mass) was shown to exist.

On the other hand, since the surprising numerical evidence of BK [4] on the existence of particle-like solutions (with hair) in four dimensional Einstein-Yang-Mills theory, there has been a lively research in the theories of gravity coupled to non-Abelian gauge theories. A nice account of developments and references in the subject is summarized in the review article [5]. More recently new monopole and dyon solutions were found in [6].

In this paper our intention is to study three dimensional Euclidean gravity coupled to Yang-Mills-Higgs theory. Deser's [7] earlier work in the study of Einstein-Yang-Mills theory (no Higgs) shows that there are no static solutions in this theory. In this paper we add a Higgs field in the adjoint representation of the group $SO(3)$ which is spontaneously broken down to $U(1)$ and we study the effect of gravity on the 't Hooft-Polyakov instantons in the BPS limit (the limit of vanishing self-interaction for the Higgs field). We show that one obtains the Bogomol'nyi type first order equations for the Higgs and the gauge field as in the flat space limit. There are exact solutions to the equations of motion which do not have flat space analogs (limits) but these solutions are not of finite action. We find the numerical solutions of finite action which reduce to the BPS instantons in flat space.

The reader might wonder if a spontaneously broken, $SO(3)$ down to $U(1)$, gauge theory is expected to be any different from the Einstein-Maxwell theory in which a BTZ solution was found. In the four dimensional context it [8] was shown that, on the contrary to the initial expectation, the spontaneously broken gauge theory allowed black holes with "non-trivial" hair which do not exist in Einstein-Maxwell theory. So, in principle, we do not have a strong reason to belive that the spontaneously broken theory, in three dimensions, will only have a BTZ type solution with only three types of hair. Although this question along with the question of non-Abelian hair of a BTZ black hole are extremely interesting we do not attempt these in this paper. Our immediate interest, as stated in the previous paragraph, is to explore what happens to the gauge theory instantons if gravity (with cosmological constant) is turned on. In section 2 we present the model in which we will search for an answer to this problem. In section 3 we will find numerical solutions and show that the action is independent of the choice of coordinate. In section 4 we will present an exact, singular solution to the equations of motion and finally in Section 5 we will make some concluding remarks.

2 The Model

We will work in the Euclidean space and in the first order formalism of gravity in terms of the dreibein and the spin connection. Achúcarro-Townsend [9] and Witten [10] showed that three dimensional Einstein-Hilbert action with zero cosmological constant is equivalent to Chern-Simons theory with the gauge group $ISO(3)$. In the theories with non-zero cosmological constant one can simply generalize this to $SO(4)$ or $SO(3,1)$ depending on the sign of the cosmological constant. Witten also realized that depending on the choice of the quadratic Killing-form one obtains two classically equivalent actions for gravity.

The standard action is the following

$$S_G = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\epsilon^{ijk}\Big\{2\, e^a{}_i\,\partial_j\,\omega^a{}_k + \epsilon^{abc}\, e^a{}_i\,\omega^b{}_j\,\omega^c{}_k + \frac{\lambda}{3}\epsilon^{abc}\, e^a{}_i\, e^b{}_j\, e^c{}_k\Big\} \tag{1}$$

This action is real in the Euclidean space and is equivalent to Einstein-Hilbert theory if the dreibein is invertible. Another action, which in Minkowski space is also equivalent to the Einstein-Hilbert action, often coined as "exotic action" is:

$$\begin{aligned} S_{CS} = \quad & -\frac{ik}{16\pi G}\int_{\mathcal{M}} d^3x\epsilon^{ijk}\Big\{\omega^a{}_i\,\partial_j\,\omega^a{}_k + \frac{1}{3}\epsilon^{abc}\,\omega^a{}_i\,\omega^b{}_j\,\omega^c{}_k + \lambda(e^a{}_i\partial_j\, e^a{}_k \\ & +\epsilon^{abc}\,\omega^a{}_i\, e^b{}_j\, e^c{}_k)\Big\} \end{aligned} \tag{2}$$

Assuming that the fields are real, which we do for our analysis, this action is completely imaginary because it is Wick rotated from Minkowski space. Both of these actions are considered in the path integral formulation of gravity. These actions are important geometric invariants of three manifolds; namely, they are the "volume" and the "Chern-Simons" invariants respectively. In this paper we consider both of them together. So including the Higgs and the Yang-Mills terms our full action becomes

$$S = S_G + S_{CS} + S_{YM} + S_H \tag{3}$$

where the Yang-Mills and the Higgs actions are

$$S_{YM} = -\frac{1}{4e^2} \int d^3x \sqrt{|g|} g^{ij} g^{kl} F^a_{ik} F^a_{jl} \tag{4}$$

$$S_H = -\frac{1}{e^2} \int d^3x \sqrt{|g|} \{ \frac{1}{2} g^{ij} D_i h^a D_j h^a + \frac{\nu}{6!} (h^2 - h_0^2)^3 \} \tag{5}$$

The higgs field is in the adjoint representation of $SO(3)$ and the covariant derivative is $D_i h^a = \partial_i h^a + \epsilon^{abc} A^b_i h^c$. Hence, F^a_{ij} has no gauge coupling in it.

Let us denote the dimensions of the fields and the parameters in the theory.

$$\begin{aligned} &[e^2] = M, \qquad [G] = M^{-1}, \qquad [\lambda] = M^2, \qquad [k] = M^{-1} \\ &[e^a{}_j] = M^0, \qquad [h] = M, \qquad [\omega^a{}_j] = M, \qquad [\nu] = M^{-2} \end{aligned} \tag{6}$$

We will exclusively work in the BPS limit where $\nu = 0$. The indices (a, b, c) denote the tangent space and (i, j, k) denote the manifold coordinates. The metrics , η_{ab} and g_{ij} have Euclidean signature. $\lambda < 0$ corresponds to the de-Sitter and $\lambda > 0$ to the anti-de-Sitter space. The "dual" Riemann tensor can be defined to be $R^a{}_{kj} = \partial_k \omega^a{}_j - \partial_j \omega^a{}_k + \epsilon^a{}_{bc} \omega^b{}_k \omega^c{}_j$. The relation between the Ricci tensor and the dual Riemann tensor is $R_{ij} = e^a_i E^k_b \epsilon_{abc} R^c{}_{jk}$, where E^k_b is the inverse of the dreiben. In the absence of gauge fields Einsteins equations, $R_{ij} = -2\lambda g_{ij}$, imply the scalar curvature to be $R = -6\lambda$.

We employ the well known spherically symmetric ansatz for all the fields in the theory.

$$e^a{}_j(\vec{x}) = \frac{G}{r}\left[-\epsilon^a{}_{jk}\,\hat{x}^k\,\phi_1 + \delta^a{}_j\,\phi_2 + (rA - \phi_2)\,\hat{x}^a\hat{x}_j\right] \tag{7}$$

$$w^a{}_j(\vec{x}) = \frac{1}{r}\left[\epsilon^a{}_{jk}\,\hat{x}^k\,(1-\psi_1) + \delta^a{}_j\psi_2 + (rB - \psi_2)\hat{x}^a\hat{x}_j\right] \tag{8}$$

$$A^a_j(\vec{x}) = \frac{1}{r}\left[\epsilon^a{}_{jk}\,\hat{x}^k\,(1-\varphi_1) + \delta^a{}_j\varphi_2 + (rD - \varphi_2)\hat{x}^a\hat{x}_j\right] \tag{9}$$

$$h^a(\vec{x}) = \hat{x}^a h(r) \tag{10}$$

The functions A, B, ϕ_i ,D, φ_i and ψ_i depend on r only. The meaning of r should be clear from $r^2 = \eta_{ij}x^ix^j$. We have chosen the dreiben to be dimensionless and the first term of the dreiben is chosen in a way which will yield better looking equtions.

The metric on the manifold can be recovered by from the dreibein through the relation $g_{ij} = \eta_{ab}e^a{}_i e^b{}_j$ which yields;

$$g_{ij} = \frac{G^2}{r^2}\left\{(\phi_1^2+\phi_2^2)(\delta_{ij} - \hat{x}_i\hat{x}_j) + r^2A^2\hat{x}_i\hat{x}_j\right\} \tag{11}$$

The flat space limit $(g_{ij} = \delta_{ij})$ corresponds to $G^2(\phi_1^2+\phi_2^2) = r^2$ and $A(r)G = 1$.

The dual Riemann tensor and the non-Abelian field strength tensor can be obtained from a tedious but straightforward computation. Clearly both of them are of the same form.

$$\begin{aligned} R^a{}_{ij} \;\; &= \;\; \frac{1}{r^2}\epsilon_{ijb}\,\hat{x}^a\hat{x}^b\,(\psi_1^2+\psi_2^2-1) + \frac{1}{r}(\epsilon^a{}_{ij} - \epsilon_{ijb}\hat{x}^a\hat{x}^b)(\psi_1' + B\psi_2) \\ &\quad + (\delta^a{}_j\hat{x}_i - \delta^a{}_i\hat{x}_j)\frac{1}{r}(\psi_2' - B\psi_1) \end{aligned} \tag{12}$$

$$\begin{aligned} F^a{}_{ij} \;\; &= \;\; \frac{1}{r^2}\epsilon_{ijb}\,\hat{x}^a\hat{x}^b\,(\varphi_1^2+\varphi_2^2-1) + \frac{1}{r}(\epsilon^a{}_{ij} - \epsilon_{ijb}\hat{x}^a\hat{x}^b)(\varphi_1' + D\varphi_2) \\ &\quad + (\delta^a{}_j\hat{x}_i - \delta^a{}_i\hat{x}_j)\frac{1}{r}(\varphi_2' - D\varphi_1) \end{aligned} \tag{13}$$

Before we write down the reduced form of the action let us denote the determinant of the metric

$$\det e = \sqrt{|g|} = \frac{G^3}{r^2}|A|(\phi_1^2+\phi_2^2) \tag{14}$$

The actions reduce to the following one dimensional forms (from now on the repeated indices (a, b) take values in $(1, 2)$ and a summation is implied). The Einstein-Hilbert action is

$$S_G = -\int_0^\infty dr \left\{ \psi'_a \epsilon_{ab} \phi_b + B\psi_a\phi_a + \frac{A}{2}(\psi_a\psi_a + \lambda G^2 \phi_a\phi_a - 1) \right\} \tag{15}$$

The Chern-Simons action is

$$\begin{aligned} S_{CS} = \quad & -ik \int_0^\infty dr \left\{ \psi'_a \epsilon_{ab} \psi_b + \psi'_2 + B(\psi_a\psi_a + \lambda G^2 \phi_a\phi_a - 1) \right. \\ & \left. +\lambda G^2(\phi'_a \epsilon_{ab} \phi_b + 2A\phi_a\psi_a) \right\} \end{aligned} \tag{16}$$

The Yang-Mills action is

$$S_{YM} = -\frac{2\pi}{e^2 G} \int_0^\infty dr \frac{1}{|A|\phi_a\phi_a} \left\{ A^2(\varphi_a\varphi_a - 1)^2 + 2\phi_a\phi_a(\varphi'_b\varphi'_b + 2D\epsilon_{ab}\varphi'_a\varphi_b + D^2\varphi_a\varphi_a) \right\} \tag{17}$$

In the BPS limit ($\nu = 0$) and in the broken phase ($h_0 \neq 0$) the Higgs term is

$$S_H = -\frac{2\pi G}{e^2} \int_0^R dr \left\{ \frac{1}{|A|}\phi_a\phi_a h'^2 + 2h^2|A|\varphi_a\varphi_a \right\} \tag{18}$$

From here we will assume that A(r) is positive so that we may drop the absolute value sign. We will see that this requirement is satisfied in our solution.

We are interested both in the singular and the non-singular solutions. For the case of finite action and non-singular solutions the boundary conditions for the gauge and Higgs sector follow as

$$\varphi_1(0) = 1, \quad \varphi_2(0) = 0 \qquad \varphi_1(\infty) = \varphi_2(\infty) = 0 \tag{19}$$

$$h(0) = 0 \qquad h(\infty) = h_0 \qquad D(\infty) = 0 \tag{20}$$

For the gravity part non-singular dreiben and the spin connection at the origin require

$$\phi_1(0) = \phi_2(0) = 0 \qquad \psi_1(0) = 1, \quad \psi_2(0) = 0 \tag{21}$$

The equations of motion of the full theory are

$$\delta B : \qquad \psi_a\phi_a + ik(\psi_a\psi_a + \lambda G^2\phi_a\phi_a - 1) = 0 \tag{22}$$

$$\delta\psi : \qquad \epsilon_{ab}\phi'_b - B\phi_a - A\psi_a + ik(2\epsilon_{ab}\psi'_a + 2B\psi_b + 2A\lambda G^2\phi_b) = 0 \tag{23}$$

$$\delta D : \qquad \epsilon_{ab}\varphi'_a\varphi_b + D\varphi_a\varphi_a = 0 \tag{24}$$

$$\delta h : \qquad \left\{\frac{\phi_a\phi_a h'}{A}\right\}' - 2hA\varphi_a\varphi_a = 0 \tag{25}$$

$$\begin{aligned}\delta\phi : \qquad & \epsilon_{ab}\psi'_a + B\psi_b + A\lambda G^2\phi_b - ik\lambda G^2(2\epsilon_{ab}\phi'_a - 2B\phi_b - 2A\psi_b) \\ & -\frac{4\pi}{e^2 G}\frac{\phi_b A}{(\phi_c\phi_c)^2}(\varphi_a\varphi_a - 1)^2 + \frac{4\pi G}{e^2}\frac{\phi_b h'^2}{A} = 0\end{aligned} \tag{26}$$

$$\delta\varphi : \qquad \left\{\frac{\varphi'_a + D\epsilon_{ab}\varphi_b}{A}\right\}' - \frac{\varphi_a A}{\phi_a\phi_a}(\varphi_a\varphi_a - 1) - \frac{D}{A}(D\varphi_a + \epsilon_{ba}\varphi'_b) - G^2h^2A\varphi_a = 0 \tag{27}$$

$$\begin{aligned}\delta A : \qquad & \psi_a\psi_a + \lambda G^2\phi_a\phi_a - 1 + 4ik\lambda G^2\phi_a\psi_a + \frac{8\pi G}{e^2}h^2\varphi_a\varphi_a + \frac{4\pi}{Ge^2}\frac{(\varphi_a\varphi_a - 1)^2}{\phi_a\phi_a} \\ & -\frac{4\pi}{Ge^2}\frac{1}{A^2}\left\{2\varphi'_a\varphi'_a + 4D\epsilon_{ab}\varphi'_a\varphi_b + 2D^2\varphi_a\varphi_a + G^2h'^2\phi_a\phi_a\right\} = 0\end{aligned} \tag{28}$$

In general, because of the Chern-Simons term, solutions to the equations of motion will be complex. Complex dreiben and spin connection, however, will change the geometry drastically. For example, the notion of a positive definite metric will be lost. Hence, we restrict ourselves to the real solutions only. This means that the Chern-Simons term decouples from the rest. It is clear that the equations of motion for the Chern-Simons gravity are exactly the equations one gets for Einstein-Hilbert gravity without the matter fields. This fact is no secret because we know that at the classical level Einstein-Hilbert theory is equivalent to Chern-Simons theory of gravity. In this way we have obtained a nice system where we can try to analyze the effect of gravity on three dimensional 't Hooft-Polyakov Instantons. It is clear that gravity itself is not disturbed by the instantons because of the Chern-Simons term. Using these facts we take on the job of obtaining solutions to the equations in the next section.

3 Solutions of the Equations of Motion

As already stated, we are only looking for real solutions, hence the Chern-Simons term yields the following equations:

$$\epsilon_{ab}\phi_b' - A\psi_a - B\phi_a = 0 \tag{29}$$

$$\epsilon_{ab}\psi_b' - B\psi_a - \lambda G^2 A\phi_a = 0 \tag{30}$$

$$\psi_a\psi_a + \lambda G^2\phi_a\phi_a - 1 = 0 \tag{31}$$

$$\phi_a\psi_a = 0 \tag{32}$$

$$\tag{33}$$

The general solutions of these equations, compatible with the regularity conditions at the origin, were given in [11]

$$\psi_1 = \frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\cos\Omega(r) \qquad \psi_2 = \frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\sin\Omega(r) \tag{34}$$

$$\phi_1 = f(r)\frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\sin\Omega(r) \qquad \phi_2 = -f(r)\frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\cos\Omega(r) \tag{35}$$

$$A = -\frac{f(r)'}{1+\lambda G^2 f^2(r)} \qquad B = \Omega'(r) \tag{36}$$

$f(r)$ and $\Omega(r)$ are arbitrary functions which should satisfy the following conditions.

$$\Omega(0) = 2n\pi, \qquad f(0) = 0 \tag{37}$$

The line element in the polar coordinates takes the following form.

$$(ds)^2 = \frac{G^2}{1+\lambda G^2 f^2}\left\{ f^2 d\Omega_2 + \frac{1}{1+\lambda G^2 f^2}(df)^2 \right\} \tag{38}$$

Recall the difference between four and three dimensional gravity. In the four dimensional case, to be specific, if one takes the spherically symmetric solution, even after a choice of the coordinates, there is one more function left which is to be determined by the Einstein's equations of motion. In the three dimensional case since there are no nontrivial local degrees of freedom one determines the metric completely by a choice of gauge. Of course in all this construction we assume that we are not changing the global structure of the manifold by a choice of coordinates. Our choice of $f(r)$ will be given after we prove that the instanton solution does not depend on the choice of coordinates.

Using the solutions of the Chern-Simons equations of motion to simplify the equations for the Higgs and Yang-Mills fields, we can see that the resulting relations are still somewhat complicated. To see the solution more clearly one can make certain choices of gauges. For example a look at the action will reveal that the unbroken $U(1)$ acts in a way that keeps the following complex function invariant

$$\eta(r) = (\varphi_1 + i\varphi_2)e^{-i\int^r D(r')dr'} \tag{39}$$

If the Yang-Mills action is written in terms of $\eta(r)$ obviously none of the functions $(\varphi_1, \varphi_2, D)$ will appear in the action. So we can choose a gauge (the singular gauge) in which $\varphi_2 = D = 0$. Denoting $\varphi_1 = \varphi$, the remaining *independent* equations read as

$$h' = -\frac{A}{G(\phi_a \phi_a)}(\varphi^2 - 1) \tag{40}$$

$$h = -\frac{1}{GA}\frac{\varphi'}{\varphi} \tag{41}$$

These are the Bogomol'nyi type first order equations for the gravitating instanton. These equations reduce to the well known exactly solvable equations in the flat space limit.

Before we choose our coordinate function $f(r)$ let us show that instanton action is independent of this choice. We write the action in the following form:

$$\begin{aligned} S_{YM} + S_H = \quad & -\frac{4\pi}{e^2 G} \int dr \Big\{ \frac{1}{A}(\varphi' + GAh\varphi)^2 - 2G\varphi'\varphi h \\ & + \frac{G^2}{2A}\phi_a\phi_a (h' + \frac{A}{G\phi_a\phi_a(\varphi^2 - 1)})^2 - Gh'(\varphi^2 - 1) \Big\} \end{aligned} \tag{42}$$

If the equations of motion are satisfied the integrand becomes a full derivative and after integration one obtains

$$S_{Instanton} = -\frac{4\pi h(\infty)}{e^2} \tag{43}$$

This result does not explicitly dependent on the cosmological constant. It, however, can be seen from the numerical solutions that $h(\infty)$ depends on the cosmological constant.

Let us choose $f(r) = -r/G$. Then using the solutions for the Chern-Simons term, equations (40) and (41) become:

$$h'(r) = -\frac{1}{r^2}(\varphi^2(r) - 1) \tag{44}$$

$$\varphi'(r) = -\frac{1}{(1 + \lambda r^2)} h(r)\varphi(r) \tag{45}$$

In the flat space limit ($\lambda = 0$) one has the well-known BPS solution

$$\varphi(r) = \frac{h_0 r}{\sinh(h_0 r)}; \qquad h(r) = -\frac{1}{r} + h_0 \coth(h_0 r) \tag{46}$$

where $h(\infty) = h_0$. For non-zero λ the solutions can be obtained numerically and they are plotted in figure 1 and figure 2. For any positive value of λ there is a solution. Non-zero λ solutions take values between the BPS instanton solution, (46), and the vacuum solution ($h(r) = 0, \varphi(r) = 1$). For very large values of λ the solution approaches the trivial vacuum solution. For negative values of λ , de sitter case with our conventions, the existence of the horizon introduces singularities and there are no finite action solutions.

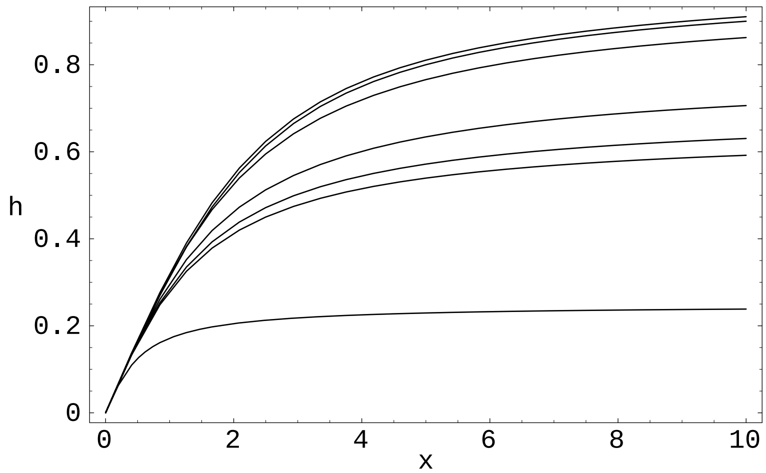

Figure 1: Higgs field $h(x)$ is shown for variuos values of λ. $h(\infty)$ approaches to zero (vacuum solution) if λ is increased. The above values, starting from the top correspond to $\lambda = 0$,$\lambda = 4.10^{-3}$,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

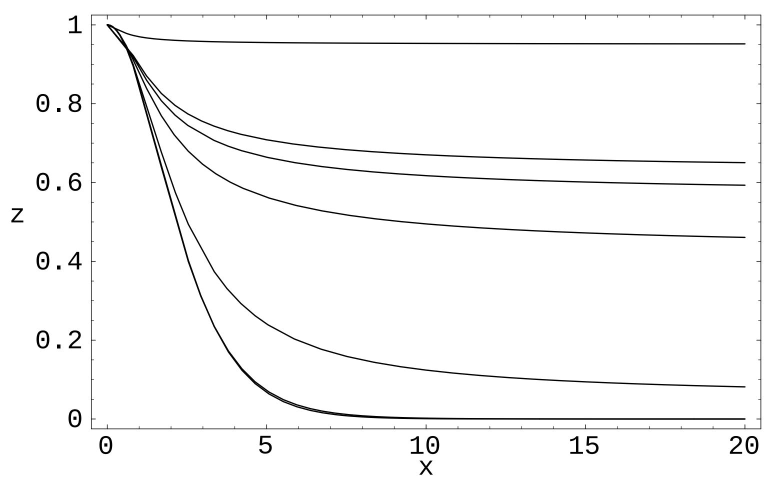

Figure 2: Non-zero component of the gauge field $\varphi(x)$ is shown for variuos values of λ. $\varphi(\infty)$ approaches to 1 (vacuum solution) if λ is increased. The above values, starting from the bottom correspond to $\lambda = 0, 4.10^{-3}$,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

4 Singular Solutions

In this section we would like to point out an exact solution to equations (40) and (41). These solutions are singular and have no non-trivial limits in the flat space.

$$\varphi(r) = -\sqrt{\lambda} G f(r), \qquad h(r) = \frac{1 + \lambda G^2 f(r)^2}{f(r) G} \tag{47}$$

Earlier we have shown that we require $f(0) = 0$ for non-singular dreibein. So, for reasonable coordinates, (47), is a singular solution and the action is infinite. For definiteness let us rewrite these solutions in the $f(r) = -r/G$ coordinates.

$$\varphi(r) = \sqrt{\lambda} r, \qquad h(r) = -(\frac{1}{r} + \lambda r) \tag{48}$$

These solutions are not gauge copies of the trivial vacuum solutions.

5 Conclusion

We have shown that when three dimensional Euclidean gravity is coupled to Yang-Mills and Higgs fields the equations of motion reduce to first order equations of the Bogomol'nyi type. We found singular and regular solutions. Our main result is that there are finite action solutions for any positive semi-definite value of the cosmological constant. Depending on the numerical value of the cosmological constant these solutions take values between the BPS solution and the trivial vacuum solution. The action can be calculated exactly and is given by (43). Finite actions solutions are stable and one can define a topological charge which is the magnetic charge. This is done following 't Hooft's definition of an Abelian field strength out side the instanton core.

In addition to the Yang-Mills term, if a Chern-Simons term is added for the gauge sector one should look for the complex gauge configurations as it was pointed out in [12]. In this case one cannot use the singular gauge as the Chern-Simons term trivially vanishes. The theory will have the following additional action

$$S = -\frac{i\kappa}{e^2} \int d^3x \epsilon^{\mu\nu\lambda} \mathrm{tr}\Big(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \Big) \tag{49}$$

Using the symmetric ansatz one gets

$$S = \frac{4i\pi\kappa}{e^2}\int_0^\infty \left[\epsilon_{ab}\varphi_a'\varphi_b + D(\varphi_a\varphi_a - 1) - \varphi_2'\right] \tag{50}$$

Clearly this term vanishes for the singular gauge which is too restrictive. In some other gauges (i.e $D = 0$ and $\varphi_2 \neq 0$) we expect complex solutions. For the case of Einstein-Maxwell-Chern-Simons theory with a Lorentzian metric we refer the reader to [13] where self-dual solutions were established.

In the context of a non-supersymmetric theory (like the one we dealt with in this paper) h_0 and λ are given to define the theory. So one changes the theory if these parameters are changed at the classical level. So our results mainly mean that there are finite action instanton solutions for those theories which have suitable pairs of λ and h_0. On the other hand if our theory is considered as a bosonic part of a Supergravity theory where a moduli space (for the Higgs field) exists then for given λ there there a many solutions.

6 Acknowledgements

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